

ALMOST INVARIANCE OF DISTRIBUTIONS FOR RANDOM WALKS ON GROUPS

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ABSTRACT. We study the neighborhoods of a typical point Z_n visited at n -th step of a random walk, determined by the condition that the transition probabilities stay close to $\mu^{*n}(Z_n)$. If such neighborhood contains a ball of radius $C\sqrt{n}$, we say that the random walk has almost invariant transition probabilities. We prove that simple random walks on wreath products of \mathbb{Z} with finite groups have almost invariant distributions. A weaker version of almost invariance implies a necessary condition of Ozawa's criterion for the property H_{FD} . We define and study the radius of almost invariance, we estimate this radius for random walks on iterated wreath products and show this radius can be asymptotically strictly smaller than $n/L(n)$, where $L(n)$ denotes the drift function of the random walk. We show that the radius of individual almost invariance of a simple random walk on the wreath product of \mathbb{Z}^2 with a finite group is asymptotically strictly larger than $n/L(n)$. Finally, we show the existence of groups such that the radius of almost invariance is smaller than a given function, but remains unbounded. We also discuss possible limiting distribution of ratios of transition probabilities on non almost invariant scales.

1. INTRODUCTION

Given a sequence of functions $f_n : G \rightarrow \mathbb{R}$, a sequence of probability measures μ_n , and a sequence of subsets $\Omega_n \subset G$, we can consider the ratios $f_n(xg_n)/f_n(x)$, where $g_n \in \Omega_n$ and ask whether this ratio is close to one for x chosen with μ_n probability close to one. If the latter happens for Ω_n being the ball of radius $r(n)$, centered at the identity, we say that f_n is almost invariant on the scale $r(n)$.

Otherwise, if this is not the case, we can ask whether $f_n(xg_n)/f_n(x)$ is close, with probability one with respect to μ_n , to a given probability distribution?

In this paper, we study the case when both f_n and μ_n are n step transition probabilities of a random walk (G, μ) . Here G is some finitely generated group and μ is a probability measure on μ , $f_n(g) = \mu_n(g) = \mu^{*n}(g)$.

Definition 1.1. Let G be a group, generated by a finite set S . Random walk (G, μ) has *almost invariant distributions* on the scale $r(n)$, if for all $\epsilon > 0$ and all n there exists a subset $V_{\epsilon,n} \subset G$, satisfying $\mu^{*n}(V_n) \geq 1 - \epsilon$, such that the following holds. For any sequences $h_n \in V_n$ and $g_n \in G$, such that $l_{G,S}(g_n) \leq r(n)$ it holds $|\mu^{*n}(h_n g_n)/\mu^{*n}(h_n) - 1| \leq \epsilon$.

For example, non-degenerate aperiodic random walks on finite groups are almost invariant on any scale. The estimates of Hebisch and Saloff-Coste imply that simple aperiodic random walks on nilpotent groups are almost invariant on any scale smaller than \sqrt{n} (see Example 2.2).

Definition 1.2. Given a function $r(n)$, we say that the *radius of almost invariance* for transition probabilities of the random walk (G, μ) is asymptotically larger than $r(n)$ if μ^{*n} is almost invariant on the scale $r''(n)$ for any $r''(n)$ such that $r''(n)/r(n) \rightarrow 0$. We write

in this case $r_{\text{a.i.}}(n) \geq r(n)$. We say that the radius of almost invariance is asymptotically smaller than $r(n)$ if for any function $r'(n)$ such that $r'(n)/r(n)$ tends to infinity on some subsequence, μ^{*n} is not almost invariant on the scale $r'(n)$. We write in this case $r_{\text{a.i.}}(n) \leq r(n)$. If $r_{\text{a.i.}}(n) \geq r(n)$ and $r_{\text{a.i.}}(n) \leq r(n)$, we say that $r_{\text{a.i.}}(n) = r(n)$ is the radius of almost invariance for transition probabilities of the random walk (G, μ) .

We believe that the radius of almost invariance is at most \sqrt{n} for any simple random walk with the support generating an infinite group. We say that the random walk (G, μ) has *almost invariant transition probabilities* if it is almost invariant on the hypothetically largest possible scale: that is, if $r_{\text{a.i.}}(n) \geq \sqrt{n}$.

We would like to stress that it is essential for the questions we study in this paper that the centers of neighborhood are chosen at random with respect to μ^{*n} . Ratios of transition probabilities in the neighborhoods of a deterministic point have been studied in various situations: in the amenable case where one can study what we call *radii of almost constancy* (see the definition below and section 2.1); in the non-amenable case where one can study *ratio theorems* for transition probabilities. Ratio $\mu^{*n}(hg)/\mu^{*n}(h)$, with h chosen at random on the trajectory of the random walk (G, μ) have been studied so far only for fixed increments, according to our knowledge. In the latter case, it has been proven by Kaimanovich and Vershik [13] that closeness of such ratios to one characterizes non-triviality of the Poisson boundary.

The radius of almost invariance describes the size of a neighborhood of a typical point (chosen at random with respect to μ^{*n}) such that the transition probabilities are almost constant in such neighborhood. If instead of a random point we consider a fixed point, the size of such neighborhood can be quite different, see Subsection 2.1). The existence of a non-trivial neighborhood of the identity with almost constant decay for a symmetric random walk on G is equivalent to the amenability of G (as it follows from the ratio theorem of Avez, see Lemma 2.15), while the existence of non-trivial scale of almost invariance is equivalent to the triviality of the Poisson boundary (as it follows from the results of Kaimanovich and Vershik, mentioned above, see Proposition 2.6).

Let us say that $R(n)$ is the *radius of almost constancy* for transition probabilities of the random walk (G, μ) if there exists $C > 0$ such that $\mu^{*n}(g)/\mu^{*n}(e)$ is bounded (by C from above and $1/C$ from below) for any element g of length at most $R(n)$ and if for any increasing sequence n_i and R' such that $R'(n_i)/R(n_i)$ tends to infinity, there exists a sequence g_i , of length at most $R'(n_i)$, such that $\mu^{*n_i}(g)/\mu^{*n_i}(e)$ is not bounded. This definition is mostly interesting for symmetric measure μ , and in this case the ratio in question is bounded if it is bounded from below, for even n .

The radii of almost invariance and almost constancy of transition probabilities in the neighborhood of a fixed point are asymptotically equivalent for simple random walks on nilpotent groups. For all random walks with non-trivial boundary (as well as for various examples of random walks with trivial boundary) the radius of almost constancy in the neighborhood of a fixed point is asymptotically equal or larger than the radius of almost invariance. We show however that there exist groups such that the radius of almost invariance for simple random walks on such groups is larger than the radius of almost constancy in the neighborhood of a fixed point. This is the case for wreath product of \mathbb{Z} with a finite group, as follows from Theorem 1.3.

Recall that the *wreath product* of A and B , which we denote by $A \wr B$, is a semi-direct product of A and $\sum_A B$, where A acts on $\sum_A B$ by shifts.

A probability measure μ on the wreath product of A with B , defines a *switch-walk-switch random walk*, if $\mu = \mu_B * \mu_A * \mu_B$, where μ_A and μ_B are measures on A and B correspondingly.

In case when $A = \mathbb{Z}$ and B is a finite group, we say that μ defines a standard simple switch-walk-switch random walk on $G = A \wr B$, if $\mu = \mu_B * \mu_A * \mu_B$, where μ_A is a symmetric measure on \mathbb{Z} with its support being equal to $-1, 0, 1$ and μ_B is the equidistribution on B .

theorem1

Theorem 1.3. *Let $G = \mathbb{Z} \wr F$, F is a finite group. Let (G, μ) be a standard simple switch-walk-switch random walk on G .*

1) [Almost Invariance.] *For any ϵ there exists $c > 0$ and subsets V_n , such that $\mu^{*n}(V_n) \geq 1 - \epsilon$ and such that for any $h \in V_n$ and any $g \in B(e, c\sqrt{n})$*

$$|\mu^{*n}(hg)/\mu^{*n}(h) - 1| \leq \epsilon,$$

where $B(e, cn)$ is the ball of radius cn is some word metric of G .

2) [Invariance] *For any ϵ there exists $c > 0$ and subsets V_n , such that $\mu^{*n}(V_n) \geq 1 - \epsilon$ and such that for any $h \in V_n$ and any $g \in B(e, c\sqrt{n}) \cap G_0$*

$$\mu^{*n}(hg) = \mu^{*n}(h),$$

where $G_0 = \sum F \subset G$.

The first claim of Theorem 1.3 shows that the radius of almost invariance is $\geq \sqrt{n}$, and this implies that $G = \mathbb{Z} \wr F$ satisfies the first assumption of proposition in Section 4 of [19]. (In fact, a weaker claim that the *radius of individual almost invariance* (see Definition 1.2 in Section 2) is asymptotically larger than \sqrt{n} is sufficient for this conclusion). Since it is well known and easy to see that the wreath products of \mathbb{Z} with finite groups admit a sequence of subsets $U_n \subset B(e, CN)$ for some $C > 0$, such that $\#\partial U_n / \#U_n \leq 1/n$ (that is, they admit a *controlled Foelner sequence* in the terminology of [2]), these wreath products satisfy also the second assumption of Ozawa's proposition. Theorem 1.3 provides therefore an answer for a question of Ozawa (private communication and a remark before the proposition in Section 4 of [19]) about existence of groups of super-polynomial growth where his criterion can be applied to prove the property H_{FD} , defined by Shalom in [22]. We remind that the property H_{FD} states that any unitary representation with non trivial first reduced cohomology group of a group G on a Hilbert space contains a finitely dimensional invariant subspace. Since any group without property T of Kazhdan, in particular any amenable group, admits a unitary representation with non-trivial first reduced cohomology group ([16, 18], see also [22] and appendix in [19]), this property provides a sufficient criterion for an amenable group to admit a non-trivial virtual quotient to \mathbb{Z} .

Construction of virtual homomorphism to \mathbb{Z} is the key step in all known proofs of the Polynomial Growth theorem of [11]. We remind that the original argument by Gromov [11] uses the fact that the space, obtained as a limit of normalized word metrics of G , is locally compact, and this is never the case when the growth is super-polynomial. The proof of Kleiner [15] uses the fact that the space of μ -harmonic functions of linear growth on a group of polynomial growth is finitely dimensional, and a natural conjecture is that the space of μ -harmonic functions of linear growth on any group of super-polynomial growth is infinite dimensional. In contrast with these proofs, we show that the argument of Ozawa, obtained in his proof of Polynomial growth theorem, can be applied to some groups of super-polynomial growth.

While the assumption on the measure in Theorem 1.3 is important for the second claim, this assumption is not necessary for the first claim of the theorem. It can be shown that the

same conclusion holds for any simple random walk. Moreover, the argument in the proof of the first part of Theorem 1.3 can be applied to large classes of groups, we plan to return to this question elsewhere. We ask in this context:

Question 1.4. Let G be a group of finite Pruefer rank (e.g. polycyclic) and μ be a non-degenerate symmetric finitely supported measure on G . Is it true that the distributions of (G, μ) are almost invariant?

Recall that the drift function $L(n)$ is the mean displacement after n steps of the random walk, that is, $L(n) = \sum_{g \in G} l_S(g) \mu^{*n}(g)$.

In contrast with $L(n)$, the radii of almost invariance are defined for measures not necessarily satisfying a moment condition. The asymptotic behavior of the radii of almost invariance is related to, but not defined by the drift function.

In Proposition 4.3 we show that the almost invariance radius of a simple switch-walk-switch random walk on j times iterated wreath product of \mathbb{Z} satisfies

$$r_{\text{a.i.}}(n) \leq C n^{1/2^{i+1}} / \sqrt{\ln(n)}.$$

This provides examples of simple random walks with almost invariance radius strictly smaller than $n/L(n)$, since in this example $n/L(n) \sim n^{1/2^{i+1}}$.

In Proposition 4.7 we show that the radius of individual almost invariance (as defined in Definition 1.2) of a standard switch-walk-switch random walk on a wreath product of \mathbb{Z}^2 with a finite group is bounded from below by a radius of a ball covered by its projection to \mathbb{Z}^2 . This implies that the radius of individual almost invariance satisfies for all sufficiently large n

$$\bar{r}(n) \geq \exp(\sqrt{\ln(n)})$$

(see Corollary 4.8), and thus it is asymptotically larger than $n/L(n)$, since in these examples $n/L(n) \sim \ln(n)$.

Question 1.5. Is it true that any group of finite Pruefer rank satisfies the property H_{FD} of Shalom? Does any group such that the drift of a simple random walk on this group is asymptotically equivalent to \sqrt{n} satisfy the property H_{FD} of Shalom?

Since groups with property H_{FD} admit a finite index subgroup with infinite Abelianisation ([22]), the positive answer to the question above would imply two conjectures below.

Conjecture 1.6. *The drift function of any simple random walk on an infinite finitely generated simple group is strictly larger than \sqrt{n} .*

Conjecture 1.7. *The drift function of any simple random walk on an infinite finitely generated torsion group is strictly larger than \sqrt{n} .*

It is proven in [7], Corollary 1 that the drift of simple random walks on the first Grigorchuk group satisfies $L(n) \geq n^\gamma$, for some $\gamma > 1/2$ and infinitely many n . It can be shown that the drift of torsion groups, e.g. some of Grigorchuk groups from [10] that are close on some scales to solvable groups, can be arbitrary close to \sqrt{n} on an infinite subsequence (moreover, on a sequence of arbitrarily quickly growing intervals).

It seems interesting to describe smallest possible drifts for simple random walks on infinite simple groups and on infinite torsion groups.

The paper has the following structure. In Section 2 we describe first examples and basic properties of almost invariance and radii of almost invariance. In Proposition 2.6 we describe the relation with the Poisson boundary. We define a weaker notion of *individual almost invariance* on a given scale, and in Corollary 2.11 we give a lower bound for the

radii of individual almost invariance in terms of entropy of random walks. Lemma 2.12 is essential for optimal lower bounds for radii of individual almost invariance for various groups: the idea is that in order to prove almost invariance for μ^{*n} , it is sufficient to chose a sequence of events α_n on the space of the trajectories of the random walk (G, μ) , such that the probabilities of α_n are close to one, and to prove the almost invariance for the sequence of conditional measures $\mu_n = \mu^{*n}(g)|_{\alpha_n}$. A similar ideas are used in the proof of Theorem 1.3 and Proposition 4.7 to get a stronger conclusion and to bound from below (non-individual) radius of almost invariance. In Subsection 2.1 we compare the radius of almost invariance with that of almost constancy of transition probabilities.

In Section 3 we prove Theorem 1.3.

In Section 4 we estimate the radii of almost invariance for distributions of simple random walks on iterated wreath products with \mathbb{Z} and wreath products of \mathbb{Z}^2 with finite groups.

In Section 5 we prove the following proposition that shows that invariance radii can be arbitrary small.

Proposition 1.8. *For any $F(n)$ tending to infinity there exists a finitely generated group G such that the almost invariance radius of any simple random walk on G is not bounded and $r_{a.i.}(n) \leq F(n)$.*

In the last Section we show that, though no group with non-trivial boundary can admit a non-trivial scale for almost invariance, it may happen that $\mu^{*n}(hg_n)/\mu^{*n}(h)$, normalized by a constant C_g , admits a (non-constant) limit distribution, see Example 6.2 for the case of standard simple random walks on a free non-Abelian group.

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2. PRELIMINARIES

Recall that the period of a probability measure μ on a group G is the greatest common divisors of $K \geq 0$ such that $\mu^{*K}(e) > 0$. A probability measure μ is called *aperiodic* if its period is equal to one. For example, if $\mu(e) > 0$ then μ is aperiodic. Observe that for any symmetric measure on a group it holds $\mu^{*2}(e) > 0$, and hence a symmetric measure μ on a group G is aperiodic if and only if there exists an odd integer K such that $\mu^{*K}(e) \neq 0$.

Example 2.1 (Finite groups). *Let F be a finite group, and μ be a non-degenerate aperiodic random walk on F . For any sequence $r(n)$ the distributions of (G, μ) are almost invariant on the scale $r(n)$.*

Example 2.2 (Nilpotent groups). *Let G be a finitely generated virtually nilpotent group, μ be a finitely supported symmetric non-degenerate aperiodic random walk on F . Then the distributions of the random walk (G, μ) are almost invariant on the scale $r(n)$ if $r(n)/\sqrt{n} \rightarrow 0$.*

Proof. Follows from the "gradient estimate" of Hebisch and Saloff-Coste (see line 2 on p.675 and (14) of Theorem 5.1 in [12]), who prove, under assumption $\mu(e) > 0$ that

$$\mu^{*n}(g) - \mu^{*n}(gs) \leq C'n^{-(D+1)/2} \exp(-l_S^2(x)/CN),$$

where S is a finite generating set of G (see first line on page 675, as well as (14) of Theorem 5.1).

This estimate shows that if $l_S(g) \leq K\sqrt{n}$, then $\mu^{*n}(g) - \mu^{*n}(gs) \leq C''n^{-(D+1)/2}$ for some $C'' > 0$ and any $s \in S$. Therefore, $\mu^{*n}(g) - \mu^{*n}(gh) \leq C''n^{-D/2}l(h)/\sqrt{n} \leq$

$C''K\epsilon n^{-D/2}$ for any $g : l_S(g) \leq K\sqrt{n}$ and $h : l_S(h) \leq \epsilon\sqrt{n}$. Since $\mu^{*n}(g) \geq C_0 n^{-D/2}$ for some $C_0 > 0$ depending on K and any $g : l_S(g) \leq K\sqrt{n}$ ([12]), we conclude that $|\mu^{*n}(g)/\mu^{*n}(gh) - 1| \leq \epsilon K''$ for some K'' depending on K , and all $g : l_S(g) \leq K\sqrt{n}$ and $h : l_S(h) \leq \epsilon\sqrt{n}$.

Therefore, for each K there exists $\delta > 0$ such that the following holds. Let $r(n)$ satisfies $r(n)/\sqrt{n} \rightarrow 0$. Consider the ball $V_n = B_{G,S}(e, K\sqrt{n})$. Then for any $h_n \in V_n$ and any $g_n : l_{G,S}(g_n) \leq r(n)$ it holds

$$|\mu^{*n}(h_n g_n)/\mu^{*n}(h_n) - 1| \leq \delta.$$

Since the statement holds true for measures with $\mu(e) \neq e$, it is clear it holds also for any aperiodic measure.

Definition 1.1 of almost invariance on some scale requires the existence of one single set V_n for all increments of length at most $r(n)$. In the Definition below we give a weaker version of this notion, where V_n can depend on the sequence g_n satisfying $l_{G,S}(g_n) \leq r(n)$.

Definition 2.3. Let G be a group, generated by a finite set S . Random walk (G, μ) has *individually almost invariant distributions* on the scale $r(n)$, if for all $\epsilon > 0$ and all g_n satisfying $l_{G,S}(g_n) \leq r(n)$ there exist subsets $V_{\epsilon,n} \subset G$, such that $\mu^{*n}(V_{\epsilon,n}) \geq 1 - \epsilon$ and such that for any sequences $h_n \in V_{\epsilon,n}$ it holds $|\mu^{*n}(h_n g_n)/\mu^{*n}(h_n) - 1| \leq \epsilon$.

If in the definition above we replace $|\mu^{*n}(h_n g_n)/\mu^{*n}(h_n) - 1| \leq \epsilon$ by the condition that $\mu^{*n}(h_n g_n)/\mu^{*n}(h_n) - 1$ is bounded from below and from above by some positive constant and the condition $\mu^{*n}(V_{\epsilon,n}) \geq 1 - \epsilon$ by $\mu^{*n}(V_{\epsilon,n}) > a$ for some positive constant a and all n , then we say that the random walk has *individually almost invariant distributions weakly* on the scale $r(n)$.

Definition 2.4. Given a function $r(n)$, we say that *the radius of individual almost invariance* for transition probabilities of the random walk (G, μ) is asymptotically larger than $r(n)$ if μ^{*n} is individually almost invariant on the scale $r''(n)$ for any $r''(n)$ such that $r''(n)/r(n) \rightarrow 0$. We write in this case $\bar{r}(n) \geq r(n)$. We say that the radius of individual almost invariance is asymptotically smaller than $r(n)$ if for any function $r'(n)$ such that $r'(n)/r(n)$ tends to infinity on some subsequence, μ^{*n} is not almost invariant on the scale $r'(n)$. We write in this case $\bar{r}(n) \leq r(n)$. If $\bar{r}(n) \geq r(n)$ and $\bar{r}(n) \leq r(n)$, we say that $\bar{r}(n) = r(n)$ is the radius of almost invariance for transition probabilities of the random walk (G, μ) .

Remark 2.5 (General Markov chains). For random walks on groups, it is not important whether we consider multiplication on the left or from the right in the definition of almost invariance. For general random walks and individual sequence g_n , the definition makes sense for the multiplication on the left and there seems to be no analog of this notion in the case of multiplication on the right. However, for uniform almost invariance on a given scale, one can define this notion for the multiplication on the right as well.

Proposition 2.6 (Relation with the triviality of the Poisson boundary). *Let G be a finitely generated group and μ be a non-degenerate aperiodic measure on G . The following properties are equivalent*

- (1) *There exists a function $R(n)$, tending to infinity, such that the distributions of the random walk (G, μ) are almost invariant on the scale $R(n)$.*
- (2) *There exists a sequence $R(n)$, tending to infinity on some subsequence, such that the distributions of the random walk (G, μ) are weakly individually almost invariant distributions on the scale $R(n)$.*

(3) *The Poisson boundary of (G, μ) is trivial.*

Proof. It is clear that (1) implies (2).

Let us show that (2) implies (3). Take a random walk (G, μ) with non trivial Poisson boundary.

By the Law of 0 and 2 (see Kaimanovich, [14], Theorem 2.1, p.155), we know that for any $\epsilon > 0$ there exists probability distributions such that $\sum_{g \in G} |\nu_1 \mu^{*n}(g) - \nu_2 \mu^{*n}(g)| \geq 2 - \epsilon$ for all sufficiently large n . Observe that for any $\epsilon > 0$ we can chose ν_1 and ν_2 as above to have finite support. Observe also that if $\sum_{g \in G} |\nu_1 \mu^{*n}(g) - \nu_2 \mu^{*n}(g)| \geq 2 - \epsilon$ for some n , then for some h_1 and h_2 in the supports of ν_1 and ν_2 it holds $\sum_{g \in G} |h_1 \mu^{*n}(g) - h_2 \mu^{*n}(g)| = \sum_{g \in G} |\mu^{*n}(g) - h^{-1} h_2 \mu^{*n}(g)| \geq 2 - \epsilon$. Since $\sum_{g \in G} |\mu^{*n}(g) - h^{-1} h_2 \mu^{*n}(g)|$ is a non-increasing function in n , this implies that for any $\epsilon > 0$ there exists $h \in G$ such that $\sum_{g \in G} |\mu^{*n}(g) - h \mu^{*n}(g)| \geq 2 - \epsilon$ for all sufficiently large n .

Therefore, for any $\epsilon > 0$ and $C > 0$ there exists $h \in G$ such that for all sufficiently large n .

$$\mu^{*n}(g \in G : \mu^{*n}(gh)/\mu^{*n}(g) > C) \geq 1 - \epsilon.$$

This is in contradiction with (2).

Now we show that (3) implies (1). Since the Poisson boundary is trivial, we know from Theorem 4.2 of [13] that for any $g \in G$

$$\mu^{*n}(x \in G : |1 - \mu^{*n}(gx)/\mu^{*n}(x)| > \epsilon) \rightarrow 0,$$

as n tends to ∞ .

Take $A : \exp(An) \geq v_S(n)$ for all n , where $v_S(n)$ is the growth function of G with respect to S . Observe that for any $N_1 < N_2 < N_3 \dots \in \mathbb{N}$ there exist $M_1 < M_2 < \dots$ such that for any $n > M_i$ and any $g : l_S(g) \leq N_i$

$$\mu^{*n}(x \in G : 1 - 1/N_i < \mu^{*n}(xg)/\mu^{*n}(x) < 1 + 1/N_i) > 1 - \exp(-AN_i)/N_i.$$

Note that there exists V_i such that for all $n \geq M_i$ it holds $\mu^{*n}(V_i) \geq 1/N_i$ and

$$1 - 1/N_i < \mu^{*n}(xg)/\mu^{*n}(x) < 1 + 1/N_i$$

for all $g : l_S(g) \leq N_i$ and all $n \geq M_i$.

Take a non-decreasing function $r(n)$, tending to ∞ as r tends to ∞ such that $r(M_i) \leq N_{i-1}$ for all i . It is clear that for any ϵ , any sufficiently large n and any g of length at most $r(n)$

$$\mu^{*n}(x \in G : |1 - \mu^{*n}(gx)/\mu^{*n}(x)| > \epsilon) \leq \epsilon.$$

This completes the proof of the proposition.

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Remark 2.7. Let μ and μ' be probability measures on a space X .

1) If for some $K, \epsilon > 0$ $\mu(g)/\mu'(g) \leq K$ with probability $\geq 1 - \epsilon$ with respect to μ , then

$$\sum_{x \in X} |\mu(x) - \mu'(x)| \leq 2(\epsilon + K - 1)$$

2) If for some $K_2, \epsilon_2 > 0$ it holds $\mu(g)/\mu'(g) \geq K_2$ with probability $\geq \epsilon_2$ with respect to μ , then

$$\sum_{x \in X} |\mu(x) - \mu'(x)| \geq \epsilon_2(K - 1)$$

Proof.

1). Observe that

$$\sum_{x \in X} |\mu(x) - \mu'(x)| = 2 \sum_{x \in X: \mu(x) \geq \mu'(x)} |\mu(x) - \mu'(x)| \leq 2(\epsilon + K - 1)$$

2). The proof is straightforward.

Corollary 2.8. *The transition probabilities of the random walk (G, μ) are almost invariant with respect to a sequence g_n if and only if the total variance between convolutions of μ and shifted convolutions of μ tends to zero:*

$$\sum_{g \in G} |\mu^{*n}(g) - \mu^{*n}(gg_n)| \rightarrow 0,$$

as $n \rightarrow \infty$.

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Remark 2.9. Let S be a finite subset of G . Suppose that for any $s \in S$ it holds

$$\sum_h |(\mu^{*n}(h) - \mu^{*n}(hs))| \leq 1/F(n).$$

Then for any g in the subgroup generated by S

$$\sum_h |(\mu^{*n}(h) - \mu^{*n}(hg))| \leq l_S(g)/F(n).$$

Proof. Observe that

$$\sum_h |(\mu^{*n}(h) - \mu^{*n}(hab))| \leq \sum_h |(\mu^{*n}(h) - \mu^{*n}(ha))| + \sum_h |(\mu^{*n}(h) - \mu^{*n}(hb))|.$$

corollary1

Corollary 2.10. *Let μ be a finitely supported measure on G such that for some $F(n)$ and for any s in the support of μ*

$$\sum_h |(\mu^{*n}(h) - \mu^{*n}(hs))| \leq 1/F(n).$$

Take $f(n)$ such that $f(n)/F(n) \rightarrow 0$ as $n \rightarrow \infty$. The distributions of (G, μ) are individually almost invariant on the scale $f(n)$.

Proof. Follows from Remark 2.9 and 2) of Remark 2.7.

Recall that an *entropy* of a probability measure ν on a space X is $-\sum_{x \in X} \nu(x) \log(\nu(x))$. Given a probability measure μ on a group G denote by $H(n)$ the entropy of its n -th convolution, $H(n) = H(\mu^{*n})$.

corollaryentropy

Corollary 2.11. *Let μ be a finitely supported aperiodic measure on G and $f(n)$ be a function such that $f(n)\sqrt{H(n+1) - H(n)} \rightarrow 0$ as $n \rightarrow \infty$. The distributions of (G, μ) are individually almost invariant on the scale $f(n)$.*

Proof. For some $C > 0$ and any s in the support of μ

$$\sum_h |(\mu^{*n}(h) - \mu^{*n}(hs))| \leq C\sqrt{H(n+1) - H(n)},$$

(see 2) of Lemma 5.1 in [9]. In that lemma it is assumed that $\mu(e) > 0$ and this assumption can be easily replaced by aperiodicity). The Corollary follows therefore from Corollary 2.10.

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Lemma 2.12. *[Conditioned measures, individual almost invariance] 1) Let μ_n be a sequence of probability measures on G , $f_n : G \rightarrow \mathbb{R}_+$. Let α_n be a sequence of events of for μ_n such that $\mu_n(\alpha_n) \rightarrow 1$ as $n \rightarrow \infty$. Let μ'_n be the conditional measure $(\mu_n|_{\alpha_n})$. If f_n are (K, ϵ) are individually almost invariant with respect to μ'_n and Ω_n , then for any ϵ'*

and any sufficiently large n f_n are $K(1 + \epsilon')$, $\epsilon(1 + \epsilon')$ individually almost invariant with respect to μ_n and Ω_n .

2) Let $\mu_n = \mu^{*n}$ and let α_n be a sequence of events on G^n viewed as the space of trajectories of length n for the random walk (G, μ) such that $P[\alpha_n] \rightarrow 1$ as $n \rightarrow \infty$. Let μ'_n be the projection on the n -th coordinate of G^n of the conditional measure $(\mu_n | \alpha_n)$. If f_n are (K, ϵ) individually almost invariant with respect to μ'_n and Ω_n , then for any ϵ' and any sufficiently large n f_n are $K(1 + \epsilon')$, $\epsilon(1 + \epsilon')$ individually almost invariant with respect to μ_n and Ω_n .

Proof. 1) Since $\mu_n(\alpha_n) \rightarrow 1$, it holds $\sum_{g \in G} |\mu_n(g) - \mu'_n(g)| \rightarrow 0$. Therefore, for any $\epsilon' > 0$ and any sufficiently large n $\mu_n(g \in G : |\mu_n(g)/\mu'_n(g) - 1| \geq \epsilon') \leq \epsilon'$. Fix $h_n \in \Omega_n$. Observe also that for any $\epsilon' > 0$ and any sufficiently large n $\mu_n(g \in G : |\mu_n(gh_n)/\mu'_n(gh_n) - 1| \geq \epsilon') \leq \epsilon'$. Put $U_n^h = \{g \in G : |\mu_n(gh)/\mu'_n(gh) - 1| \leq \epsilon_1 \text{ and } |\mu_n(g)/\mu'_n(g) - 1| \leq \epsilon_1\}$. Suppose that V_n is such that $\mu_n(V_n) \geq 1 - \epsilon$ and

$$1/K \leq \mu'_n(g_n h_n)/\mu_n(g_n) \leq K$$

for $n \geq N$, all $g_n \in V_n$ and all $h_n \in \Omega_n$. Consider $\tilde{V}_n = V_n \cap U_n$. We have $\mu_n(\tilde{V}_n) \geq 1 - \epsilon - 2\epsilon_1$. Observe that for any $g_n \in \tilde{V}_n$ and any $h_n \in \Omega_n$ it holds

$$1/K(1 - \epsilon')^2 \leq \mu_n(g_n h_n)/\mu_n(g_n) \leq K(1 + \epsilon')^2$$

This implies the statement of 1).

2) Analogously to 1), we observe that $\sum_{x \in G^n} |\mu_n(x) - \mu''_n(x)| \rightarrow 0$, where μ''_n is the conditional measure for μ_n for the condition α_n . This implies that for any $\epsilon' > 0$ and any sufficiently large n it holds $\mu_n(x \in G^n : |\mu_n(x)/\mu''_n(x) - 1| \geq \epsilon_1) \leq \epsilon_1$. Therefore, the same condition holds for the projection of μ_n and μ''_n to the n -th coordinate of G^n : for any $\epsilon' > 0$ and any sufficiently large n $\mu_n(g \in G : |\mu_n(g)/\mu'_n(g) - 1| \geq \epsilon_1) \leq \epsilon_1$. Similarly to the proof of 1), this implies the statement of 2).

invariant individually

Lemma 2.13 (A general upper bound for $\bar{r}(n)$). *Let $g_n \in G$ and A_n be an event on the space of trajectories of length n . Suppose that $\mu^{*n}(A_n) > c$ for some $c > 0$ and all sufficiently large n . Suppose also that $\mu^{*n}(g_n A_n)$ tends to zero as n tends to zero. Then the distributions of (G, μ) are not weakly almost invariant with respect to g_n . In particular, the radius of individual almost invariance satisfies $\bar{r}(n) \leq l(g_n)$.*

Proof. Suppose that the distributions of (G, μ) are weakly almost invariant with respect to g_n . Then on some subsets of probability close to one the ratio $\mu^{*n}(g_n x)/\mu^{*n}(x) > C$, for some $C > 0$ and all sufficiently large n . This would in particular hold for subsets of A_n of probability at least $c/2$. This would imply that $\mu^{*n}(g_n A_n) \geq c/2C$ for all sufficiently large n .

In certain situation one gets a better upper bound for $r_{a.i.}(n)$ from using the following:

manot almost invariant

Lemma 2.14 (A general upper bound for $r_{a.i.}(n)$). *Let A_n be a sequence of events on the space of trajectories of length n for the random walk (G, μ) such that $\mu^{*n}(A_n) \rightarrow 1$. Let μ'_n be the conditioned measure μ^{*n} with respect to A_n . Suppose that for any $V_n \subset G$, with $\mu^{*n}(V_n) \rightarrow 1$ and any $K > 0$ there exist $g_n \in G$, $l(g_n) \leq r(n)$ and $h_n \in V_n$ such that $\mu'_n(g_n h_n)/\mu'_n(g_n) \geq K$. Then the same conclusion holds for μ^{*n} : for any V_n as above and any $K > 0$ there exist $g_n \in G$, $l(g_n) \leq r(n)$ such that $\mu_n(g_n h_n)/\mu_n(g_n) \geq K$. In particular, μ^{*n} are not almost invariant on the scale $r(n)$ (and hence the radius of individual almost invariance satisfies $r_{a.i.}(n) \leq r(n)$).*

Proof. Observe that since $\mu^{*n}(A_n) \rightarrow 1$ we know that for any $\epsilon > 0$ there exist N such that for all $n \geq N$ and all $g \in G$

$$\mu^{*n}(g) = P[X_n = g, A_n] + P[X_n = g, A_n \text{ does not hold}] \geq P[X_n = g, A_n] \geq (1-\epsilon)\mu'_n(g)$$

Observe also that there exists W_n , $\mu^{*n}(W_n) \rightarrow 1$ and a sequence ϵ_n , tending to zero, such that for all $g \in W_n$ $|\mu^{*n}(g)/\mu'_n(g) - 1| \leq \epsilon_n$. This implies the statement of the lemma.

fixedpoint

2.1. Comparison with the radius of almost constancy in the neighborhood of the identity.

lemmaamenable

Lemma 2.15. *Let G be a finitely generated group and μ be a symmetric non-degenerate aperiodic measure. The following conditions are equivalent*

- (1) *There exists a sequence $r(n)$, tending to infinity as n tends to infinity, such that the distributions of (G, μ) are almost constant in the neighborhood of e of radius $r(n)$.*
- (2) *There exists a sequence $r(n)$, tending to infinity as n tends to infinity, such that the distributions of (G, μ) are weakly almost constant in the neighborhood of e of radius $r(n)$.*
- (3) *G is amenable.*

Proof. First show that (2) implies (3). Suppose that G is non-amenable. There exists $c < 0$ such that for all $n > N$ it holds $\mu^{n+1}(e) \leq c\mu^{*n}(e)$.

This implies that for any $K > 0$ there exists m such that $\mu^{*(n+m)}(e) \leq \mu^{*n}(e)/K$ for all m and all sufficiently large n . Observe that $\mu^{*(n+m)}(e) = \sum_{g \in G} \mu^{*n}(g)\mu^{*m}(g^{-1})$.

Consider M such that $\mu^{*m}(B(e, M)) \geq 1/2$. Since μ is symmetric, we know that $\mu^{*n}(g) \leq \mu^{*n}(e)$ for all even n and all g .

Observe that for all sufficiently large n there exists an element h_n of length at most M such that $\mu^{*n}(h_n) \leq 2/K\mu^{*n}(e)$.

Now we show that (3) implies (1). The ratio theorem of Avez ([1]) states that for any symmetric non-degenerate measure μ on an amenable group G , such that $\mu^{*m}(e) \neq 0$ for some odd m and any $g \in G$ it holds

$$\lim_{n \rightarrow \infty} \mu^{*n}(g)/\mu^{*n}(e) = 1.$$

Chose $N(i)$ such that for any $n > N(i)$ and any g of length at most i

$$1 - 1/i < \mu^{*n}(g)/\mu^{*n}(e) < 1 + 1/i.$$

Let $r(n)$ be the maximum of i such that $N(i) < r(n)$. It is clear that for any g_n such that $l(g_n) \leq r(n)$ it holds $\mu^{*n}(g_n)/\mu^{*n}(e) \rightarrow 1$.

Finally, it is clear that 1 implies 2.

examplerevelle

Example 2.16 (Revelle, Theorem 1 [20]). *Let $G = \mathbb{Z} \wr F$, F is a finite group and μ be an aperiodic switch-walk-switch finitely supported measure on G . For any $\epsilon > 0$ there exists $c > 0$ such that for all sufficiently large n and all g of length at most $cn^{1/3}$ it holds $(1 - \epsilon) \leq \mu^{*n}(g)/\mu^{*n}(e) \leq 1$. For any $\alpha(n) : \alpha(n)/n^{1/3} \rightarrow \infty$ there exists $g_n \in G$, $l(g_n) \geq \alpha(n)$ such that $\mu^{*n}(g_n)/\mu^{*n}(e) \rightarrow 0$.*

Revelle uses the fact that the random walk is switch-walk-switch in the proof of Theorem 1 in [20] to obtain more precise description of the decay of transition probabilities in the neighborhood of the identity. The statement of Example 2.16 can be proven for any finitely supported aperiodic random walk on $\mathbb{Z} \wr F$

Remark 2.17. Let G be an amenable finitely generated group and μ be a non-degenerate symmetric finitely supported measure such that the Poisson boundary of (G, μ) is non-trivial. The radius of almost constancy of transition probabilities of (G, μ) is not trivial (that is, $r_{a.i.}(n)$ tends to infinity), while the radius of almost invariance is trivial ($R(n)$ is bounded).

While in Abelian groups $R(n)$ and $r_{a.i.}(n)$, as well as $\bar{r}(n)$ are asymptotically equivalent, the remark shows that $r_{a.i.}(n)$ could be asymptotically smaller than $R(n)$ (and not asymptotically equivalent). Theorem I.3 shows that for $G = \mathbb{Z} \wr F$ we have $r_{a.i.}(n) \sim n^{1/2}$, and since $R(n) \sim n^{1/3}$ we see that $r_{a.i.}(n)$ can be larger than $R(n)$.

3. PROOF OF THEOREM I.3

3.1. Auxiliary facts about random walks on \mathbb{Z} . For a random walk on \mathbb{Z} we denote by Min_n and Max_n the minimal and the maximal point of \mathbb{Z} , visited at least once until the moment n .

Lemma 3.1. *Consider a random walk on \mathbb{Z} defined by a symmetric finitely supported measure μ such that the support of μ is not equal to the identity. For any $\epsilon > 0$ there exists $c > 0$ such that with probability $\geq 1 - \epsilon$ with respect to μ^{*n} it holds*

$$-c\sqrt{n} \leq \text{Min}_n, \text{Max}_n \geq c\sqrt{n}$$

Proof. Recall that by a result of Erdős and Kac [4] the limit as $n \rightarrow \infty$ of the probability $P[\text{Max}_n \leq c\sqrt{n}] = C \int_{x=0}^c \exp(-t^2/2) \partial t$, for any non-degenerate μ on \mathbb{Z} with zero mean and finite second moment. Therefore, for $\epsilon > 0$ there exists $c > 0$ such that with probability $\geq 1 - \epsilon$ with respect to μ^{*n} it holds $\text{Max}_n \geq c\sqrt{n}$. Since Min_n is equal to Max_n for the trajectory, reflected at 0, it follows that for any $\epsilon > 0$ there exists $c > 0$ such that with probability $\geq 1 - \epsilon$ with respect to μ^{*n} it holds $-c\sqrt{n} \leq \text{Min}_n$.

For a random walk on \mathbb{Z} denote by I_n the interval of \mathbb{Z} visited until time instant n . Observe that if the support of μ belongs to $\{-1, 0, 1\}$, then this interval coincides with the sites visited until the time instant n .

Lemma 3.2. *Consider a simple random walk on \mathbb{Z} , reflected at 0. For any $C, \epsilon > 0$ there exist $a > 0$ such that n step transition probabilities satisfy*

$$|p_n(0, x+i)/p_n(0, x) - 1| \leq \epsilon$$

for any $x : 0 \leq x \leq C\sqrt{n}$ and any $i : 0 \leq i \leq a\sqrt{n}$.

Proof.

Observe that $\exp(-Kx^2/n)/\exp(-K(x+i)^2/n)$ is close to one whenever $0 \leq x \leq C\sqrt{n}$ and any $0 \leq i \leq a\sqrt{n}$ and that $p_n(0, 0)/p_n(0, x)$ is close to $\exp(-Kx^2/n)$ whenever $0 \leq x \leq C\sqrt{n}$ (as it follows from the Local Limit theorem, see e.g. Theorem 2.3.11 in [17]).

3.2. Proof of Theorem I.3. Take an element $z \in G$, $h = (a, f)$, $a \in \mathbb{Z}$, f is a finitely supported function on \mathbb{Z} with values in F . For such function, we denote by $I(h) = I(f)$ the minimal interval, containing the support of f . Denote by $J(h)$, the minimal interval containing a, e and $I(f)$.

Lemma 3.3. *Let μ be a symmetric non-degenerated finitely supported measure on $G = \mathbb{Z} \wr F$, F is a finite group containing at least 2 elements. Denote by X_1, X_2, \dots, X_n a trajectory of a random walk (G, μ) , $X_i = (x_i, f_i)$, $x_i \in \mathbb{A}$, $f_i : \mathbb{Z} \rightarrow F$.*

1) For any $\epsilon > 0$ there exists $C > 0$ such that $I(X_n)$ belong to the interval $[-C\sqrt{n}, C\sqrt{n}]$ with probability greater than $1 - \epsilon$.

2) For any $\epsilon > 0$ there exist $\epsilon_1 > 0$ such that $I(f)$ contains the interval $[-\epsilon_1\sqrt{n}, \epsilon_1\sqrt{n}]$ with probability greater than $1 - \epsilon$.

3) For any $\epsilon > 0$ there exist $\epsilon_1 > 0$ such that $I(f)$ contains the interval $[x_i - \epsilon_1\sqrt{n}, x_i + \epsilon_1\sqrt{n}]$ with probability greater than $1 - \epsilon$.

Proof. 1) Follows from the fact that the support of g belongs to the set, visited by the projection of the random walk to \mathbb{Z} up to the moment n and from the fact, that any finitely supported symmetric random walk on \mathbb{Z} , stays with a probability close to one (with respect to n step distribution) in a ball of radius $C\sqrt{n}$.

2) Lemma 3.1 implies that with the probability close to one, with respect to n step distribution, the projection of the random walk on \mathbb{Z} has visited some points X and Y , $X \leq -\epsilon_2\sqrt{n}$ and $Y \geq \epsilon_2\sqrt{n}$. Let S be a finite generating set of G and C_S be the maximum of $l_S(s)$, where $s \in \sup \mu$.

Suppose that the random walk has visited X, Y as above. Observe that the conditional probability that $I(X_n)$ contains some elements $-L$ and M , with $L, M \geq 1/2\epsilon_2\sqrt{n}$ is greater than $\exp(-C/C_S\epsilon_2\sqrt{n})$, for some $C > 0$, all ϵ_2 and all n .

This probability is close to one if ϵ_2 is small enough, and we have proved 2) the lemma.

Now let K_n be the maximal point of \mathbb{Z} visited by the projection of the random walk to \mathbb{Z} until the instant n , and let K'_n be the minimal point of \mathbb{Z} visited by the projection of the random walk to \mathbb{Z} until the instant n . Observe that $K_n - x_n$ (as well as $x_n - K'_n$) has the same distribution as a random walk on \mathbb{Z} , reflected at 0. This implies that $K_n - x_n \geq 2\epsilon_1\sqrt{n}$ and that $x_n - K'_n \geq 2\epsilon_1\sqrt{n}$ with probability close to one, for all sufficiently large n . Observe that at least one of the integer points in the interval $[K_n - \epsilon_1\sqrt{n}, K_n]$ belongs to the support of f_i , with probability close to one. With the same argument we conclude that at least one of the integer points in the interval $[K'_n, K'_n + \epsilon_1\sqrt{n}]$ belongs to the support of f_i , with probability close to one. This implies 3) and completes the proof of the Lemma.

Now we start by proving the second claim of the theorem.

lemmaswitch

Lemma 3.4. Let μ be a measure defining a standard simple switch-walk-switch random walk on $G = \mathbb{Z} \wr F$.

1) Suppose that $g_0 = (e, h)$ and that $z \in G$ is such that $J(g_0) \subset J(z)$. Then for any $n > 0$ $\mu^{*n}(z) = \mu^{*n}(g_0z)$.

2) Suppose that $g_0 = (e, h)$ and that $z = (x, h') \in G$ is such that $xJ(g_0) \subset J(z)$. Then for any $n > 0$ $\mu^{*n}(z) = \mu^{*n}(zg_0)$.

Proof. Take some $z = (a, f) \in \mathbb{Z} \wr F$. By assumption on the measure, the support of its projection to \mathbb{Z} is equal to $\{-1, 0, 1\}$. Observe that if the projection of the random walk (G, μ) visits $X, Y \in \mathbb{Z}$ until some time instant n , then this projection visits all integer points of the interval $[X, Y]$. Let $X_n = (a_n, f_n)$ be the trajectory of the random walk on G . Observe that if we condition the random walk to the condition that the projection to \mathbb{Z} visits some interval I , then for each integer $k \in I$ the value of $f_n(k)$ takes all values in F , and that these values are independent for all k (in this interval)

This implies that the transition probability

$$\begin{aligned} \mu^{*n}(z) &= \sum_I P[I(X_n) = I, I(z) \subset I, \text{ and } P(X_n) = a] \frac{1}{\#F\#I} = \\ &= \sum_{I: I(z) \subset I} P[I(X_n) = I \text{ and } P(X_n) = a] \frac{1}{\#F\#I} \end{aligned}$$

This implies the statement 1) and 2) of Lemma 3.4 and concludes the proof of the second claim of the theorem. lemmaswitch

Now we prove the first claim of the theorem.

Let us show that for each $\epsilon > 0$ there exists $a, C > 0$ and $V_{n,\epsilon} \subset G$, with $\mu^{*n}(V_{n,\epsilon}) \geq 1 - \epsilon$ for all sufficiently large n such that the following hold:

- 1) $\mu^{*n}(g) \geq \exp(-C\sqrt{n})$ for all sufficiently large n and any $g \in V_{n,\epsilon}$.
- 2) For all sufficiently large n , and all $g = (x, f) \in V_{n,\epsilon}$ the interval $I(f)$ contains $[x - 2a\sqrt{n}, x + 2a\sqrt{n}]$.

Indeed, observe the probability (with respect to μ^{*n}) of the set of z satisfying 1) is close to one, as follows from Lemma 3.4 combined with the fact that the projection of the random walk to \mathbb{Z} stays inside the interval $[-B'\sqrt{n}, B'\sqrt{n}]$ with positive probability, which is close to one if B' is large enough lemmaswitch

Observe also the probability (with respect to μ^{*n}) of the set of z satisfying 2) is close to one in view of statement 3) of Lemma 3.3. lemmasupportcontainsinterval

Now we estimate the ratio $\mu^{*n}(gx)/\mu^{*n}(g)$ for $g \in V_{n,\epsilon}$.

Let z be the generator of $\mathbb{Z} \subset \mathbb{Z} \wr F$. Observe that the transition probability $\mu^{*n}(g')$, conditioned to the fact that the projection of the random walk on \mathbb{Z} has visited an interval of length L is less or equal to $\exp(-C'L)$ for some $C' > 0$ and all l . In view of Lemma 3.2 this shows that lemmaswitch

$$|\mu^{*n}(gz^i)/\mu^{*n}(g) - 1| \leq \epsilon,$$

for all $g \in V_{n,\epsilon}$ and all $i \leq a\sqrt{n}$, whenever a is a small enough constant.

Now observe that 2) implies that for all sufficiently large n , all $i : -a\sqrt{n} \leq i \leq a\sqrt{n}$ and all $g \in V_{n,\epsilon}$ $J(gz^i)$ contains a ball of radius $a\sqrt{n}$, centered at the projection of gz^i to \mathbb{Z} .

Combining this observation with the second statement of Lemma we see that $\mu^{*n}(gz^i f) = \mu^{*n}(gz^i)$ for all f which has the trivial projection to \mathbb{Z} and such that $J(f) \subset [-a\sqrt{n}, \sqrt{n}]$.

Observe that any element g of the wreath product can be written as a product $g = x_0 g_0$, where g_0 has trivial projection to \mathbb{Z} and $x_0 \in \mathbb{Z}$. It is clear that for any word metric l_S on the wreath product there exists a constant $C > 0$ such that for all g the length of the elements g_0 and x_0 in their decomposition above satisfy $l_s(g_0), l_s(x_0) \leq Cl(g)$.

We see therefore that

$$|\mu^{*n}(gh)/\mu^{*n}(g) - 1| \leq \epsilon$$

for any h of word length at most $Ca\sqrt{n}$.

This completes the proof of the theorem.

4. MORE ON WREATH PRODUCTS AND ITERATED WREATH PRODUCTS

morewreath

Let C be a wreath product of A and B , and $c \in C$, $c = (a, f)$, where $f : A \rightarrow B$. Given $a \in A$, denote by $c((a))$ the value $f(a)$. Take $C = A \wr (A \wr B)$ and $a_1, a_2 \in A$. Observe that $c((a_1))$ is an element of $A \wr B$, denote by $c((a_1, a_2))$ its value at a_2 . Likewise, for i times iterated wreath product $C \wr (A \wr (A \wr \dots) \wr B) \dots$ we consider $c((a_1, \dots, a_i))$ defined inductively as the value of $c((a_1, \dots, a_{i-1}))$ at a_i . If $a_1 = a_2 = \dots = a_i e_A$, we also use the notation $c(e, i)$ for $c((a_1, \dots, a_i)) = c((e_A, \dots, e_A))$.

lemmaiterated

Lemma 4.1. *Let $G_i(H)$ be i times iterated wreath product of \mathbb{Z} with H (\mathbb{Z} is the group that acts). Consider switch-walk-switch symmetric finitely supported random walk μ_i on G_i and suppose that the drift function of the corresponding random walk on H is \sqrt{n} . Then the expected value after n steps of the random walk (G_i, μ_i)*

$$E[l(X_i((e, i))] = n^{1/2^{i+1}}.$$

Proof. Let $L_0(n)$ be the number of visits of e after n steps of a random walk on \mathbb{Z} . The expectation of $L_0(n)$ is $\sim C\sqrt{n}$. Consider the conditional event that $L_0 = \bar{L}_0$. If we condition moreover by the fact that the point of \mathbb{Z} visited by the random walk at the moment n is not equal to 0, then the value of $X_i(e)$ is obtained as a position after $2\bar{L}_0(n) - 1$ steps of a switch-walk-switch symmetric finitely supported switch-walk-switch random walk on $G_{i-1}(H)$. Otherwise, if we condition by the fact that the point of \mathbb{Z} visited by the random walk at the moment n is equal to 0, then the value of $X_i(e)$ is obtained as a position after $2\bar{L}_0(n)$ steps of the above mentioned switch-walk-switch symmetric finitely supported switch-walk-switch random walk on $G_{i-1}(H)$.

Observe that \sqrt{n} is concave, and therefore the expectation of $\sqrt{L_0(n)}$ is not greater than $\sqrt{E[L_0(n)]}$, and this implies that $E[l(X_1(e, 2))] \leq Cn^{1/4}$. Observe that there exists $a, p > 0$ such that with positive probability ($\geq p$) it holds $L_0(n) \geq a\sqrt{n}$. This implies that $E[l(X_1(e, 2))] \geq pKn^{1/4}$. Therefore, $E[l(X_1(e, 2))] \sim n^{1/4}$ and we have proved the statement of the lemma for $i = 1$.

Arguing by induction on i and using concavity of $n^{1/2^i}$, we obtain the claim of the lemma for any $i \geq 1$.

coriterated

Corollary 4.2. *Let $G_i(H)$ be i times iterated wreath product of \mathbb{Z} with H (\mathbb{Z} that acts each time). Consider switch-walk-switch symmetric finitely supported random walk μ_i on G_i and suppose that the drift function of the corresponding random walk on H is \sqrt{n} . Then the radius of individual almost invariance $\bar{r}(G_i, \mu_i)(n) \leq C/n^{1/2^{i+1}}$.*

For example, Corollary 4.2 can be applied to $H = \mathbb{Z}$ and states that simple switch-walk-switch random walks on $\mathbb{Z} \wr \mathbb{Z}$ have the radius of individual almost invariance asymptotically not larger than $n^{1/4}$, that in case of random walks on $\mathbb{Z} \wr \mathbb{Z} \wr \mathbb{Z}$ the corresponding radius of individual almost invariance is at most $n^{1/8}$ etc.

Proof. Follows from Lemma 4.1 and Lemma 2.13.

Since radius of almost invariance is less or equal to the radius of individual almost invariance, Corollary 4.2 implies in particular that $r_{\text{a.i.}}(G_i, \mu_i)(n) \leq C/n^{1/2^{i+1}}$.

This upper bound for $r_{\text{a.i.}}(G_i, \mu_i)(n)$ can be improved:

Proposition 4.3. *Let G_j be j times iterated wreath product of \mathbb{Z} with \mathbb{Z} . Consider switch-walk-switch symmetric finitely supported random walk μ_j on G_j . Then radius of almost invariance $r_{\text{a.i.}}(G_j, \mu_j)(n) \leq C \frac{n^{1/2^{j+1}}}{\sqrt{\ln(n)}}$.*

Corollary 4.4. *Under the assumption of Proposition 4.3, the radius of almost invariance of this walk is asymptotically strictly smaller than $n/L(n)$.*

We recall that $L(n)$ denotes the drift function of the random walk (G, μ) .

Proof. Follows from the fact that the drift of the standard switch walk switch random walk on G_i is asymptotically equivalent to $n^{1-1/2^{i+1}}$ ([6]). The random walk considered in [6] are "switch or walk" and the same argument works for "switch walk switch" random walks).

The assumption on the simple random walk in Corollary 4.4 is not important.

Proof of Proposition 4.3. Consider $G_1 = \mathbb{Z} \wr \mathbb{Z}$ and a symmetric finitely supported switch-walk-switch random walk μ_1 on G_1 . Let $X_i = (z_i, f_i)$ be the trajectory of the random walk (G_1, μ_1) , where $z_i \in \mathbb{Z}$ and f_i are finitely supported functions from \mathbb{Z} to \mathbb{Z} .

Take $a : 0 < a < 1/2$. Take some $n \geq 1$. We want to show that that, for some $C > 0$, with probability close to one with respect to μ_1^{*n} there exists $y \in \mathbb{Z}$ such that $f_n(y) \geq C\sqrt{n}\sqrt{\ln(n)}$. Indeed, let $t'_{i,n}$ be the number of visits of i by the projection

z_i of the random walk to \mathbb{Z} until the moment n , and let $t_{i,n} = 2t_{i,n}$ if $i \neq 0, i \neq z_i$; $t_{0,n} = 2t'_{0,n} + 1, t_{i,n} = 2t'_{0,n} + 1$ for $i = z_n$ if $z_n \neq 0$ and $t_{0,n} = 2t'_{0,n}$ if $z_n = 0$. Observe that the value $f_n(i)$ is obtained as a position of after $t_{i,n}$ steps of the random walk on \mathbb{Z} , and these random walks are independent for $i \in \mathbb{Z}$. Normalized $t'_{i,n}$ converge to the local time of Brownian motion, and hence with positive probability (close to one if ϵ is small enough) it is true that $t'_{i,n} \geq \epsilon\sqrt{n}$ for all $i \leq \epsilon n$, and in particular for all $i \leq n^a$. Therefore, $t_{i,n} \geq \epsilon\sqrt{n}$ for all $i \leq n^a$.

lemmaestimatebig

Lemma 4.5. *Let Y_i be a trajectory of a symmetric finitely supported random walk on \mathbb{Z} (with the support of the defining measure that contains at least two points). There exist $C_1, C_2 > 0$ such that for any n and any $L \geq \sqrt{n}$ the probability that $Y_n \geq L$ is greater or equal to $C_1(\sqrt{n}/L) \exp(-C_2(L^2/n))$*

Proof. Denote by μ the measure that defines our random walk on \mathbb{Z} . Using Gaussian estimates for the random walk (\mathbb{Z}, μ) we observe that probability that $Y_n \geq L$ satisfies

$$P[Y_n \geq L] = \sum_{i=L}^{\infty} \mu^{*n}(i) \geq \sum_{i=L}^{[L+n/L]} \mu^{*n}(i) \geq \sum_{i=L}^{[L+n/L]} 1/\sqrt{i} \exp(-Ki^2/n)$$

Since $L \geq \sqrt{n}$, we observe that for any $i : L \leq i \leq L + n/L$ it holds $\sqrt{i} \leq \sqrt{2L}$ and $i^2/n \leq (L + n/L)^2/n \leq (2L^2)/n + 2$. This implies that for some $C_1, C_2 > 0$

$$P[Y_n \geq L] \geq C_1(n/L)(1/\sqrt{n}) \exp(-C_2L^2/n) = C_1(\sqrt{n}/L) \exp(-C_2L^2/n),$$

and this completes the proof of the Lemma.

Lemma 4.5 implies that for a symmetric finitely supported random walk Y_i on \mathbb{Z} it holds $Y_i \geq \sqrt{i}\sqrt{\ln(i)}$ with probability $\geq 1/\sqrt{\ln(i)} \exp(-C \ln i) = (\sqrt{\ln(i)})^{-C}$.

Fix some positive a such that $a < 1/2$. Chosing positive C small enough we assure that $(\sqrt{\ln(i)})^{-C} < i^a$. In this case we can consider n^a independent random walks, making $t_j \geq cn^{1/2}$ steps, at least one of them stays at the time t_j at some point $\geq C\sqrt{n}\sqrt{\ln(n)}$. Therefore, with positive probability (which is close to one so far as C is small enough) there exist $y : 0 < y < n^a < n^{1/2}$ such that $f_n(y) \geq C\sqrt{n}\sqrt{\ln(n)}$.

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Lemma 4.6. *Consider a symmetric finitely supported random walk on \mathbb{Z} (with the support of the defining measure that contains at least two points). For any $\epsilon > 0$ there exists $K > 0$ such that for any sufficiently large n and any $T : \sqrt{n} \leq T \leq n^2/3$ it holds $\mu^{*n}(T + [Kn/T]) \leq \epsilon \mu^{*n}(T)$.*

Proof.

There exists $C_0, C_1, C_2 > 0$ such that for any $k \leq n^{2/3}$

$$C_0 1/\sqrt{n} \exp(-C_2 k^2/n) \leq \mu^{2n}(2k) \leq C_1 1/\sqrt{n} \exp(-C_2 k^2/n)$$

(see e.g. Theorem 2.3.11 and the remark after this theorem in [17]). Therefore, for any $k_1, k_2 \leq n^{2/3}$ the ratio $\mu^{2n}(2k_1)/\mu^{2n}(2k_2)$ is close up to multiplicative constant to $\exp(-C_2(k_1^2 - k_2^2)/n)$.

This implies the statement of the Lemma.

Now we return to the proof of the proposition. Fix some positive a such that $a < 1/2$. Take any function $C(n)$ tending to ∞ . Consider $g_n = (z, f) \in \mathbb{Z} \wr \mathbb{Z}$ and assume that for some y such that $|y - z| < n^a$, it holds $f(y) \geq C_1\sqrt{n}\sqrt{\ln(n)}$. Observe that for all $K > 0$ there exist $C > 0$, depending on K and C , such that the transition probabilities of the projection random walk ν on \mathbb{Z} satisfy $\nu^{*n}(x - C\sqrt{n}/\sqrt{\ln(n)})/\nu^{*n}(x) \geq K$ for any positive integer $x : l(x) \geq C_1\sqrt{n}\sqrt{\ln(n)}$. Consider $f' : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $f'(y) =$

$-C(n)\sqrt{n}/\sqrt{\ln(n)}$ and $f'(x) = 0$ for all $x \neq y$. Consider $h_n \in \mathbb{Z} \wr \mathbb{Z}$ such that $h_n = (0, f')$. Observe that the length of h_n with respect to the word metric on the standard generators of $\mathbb{Z} \wr \mathbb{Z}$ is at most $n^a + C(n)\sqrt{n}/\sqrt{\ln(n)} \leq 2C(n)\sqrt{n}/\sqrt{\ln(n)}$. Take a sequence δ_n , tending to 0, such that $\delta(n)/C(n) \rightarrow \infty$. and let A_n be the event on the space of trajectories of length n of (G, μ) for which the number of visits of y by the projected random walk on \mathbb{Z} is greater or equal than $\delta_i\sqrt{n}$. Let μ'_n be the conditional measure of μ^{*n} with respect to A_n .

Observe also that Lemma 4.6 implies that $\mu'_n(g_n h_n)/\mu'_n(g_n) \geq K$. Taking in account Lemma 2.14, this implies that the random walk (G, μ_1) is not almost invariant on the scale $C(n)n^{1/4}/\sqrt{\ln(n)}$ (for none of the the function $C(n)$ tending to infinity).

Therefore, the almost invariance radius for the random walk on $G_1 = \mathbb{Z} \wr \mathbb{Z}$ satisfies $r(G_1, \mu_1)(n) \leq C/(n^{1/4}/\sqrt{\ln(n)})$, and this completes the proof of Proposition for $G_1 = \mathbb{Z} \wr \mathbb{Z}$.

To prove the proposition for G_j for any $j \geq 1$, consider the value $\phi_{j,n} = f_n(\underbrace{e, e, \dots, e}_{j-1}) \in \mathbb{Z} \wr \mathbb{Z}$.

Observe that f_n is obtained as a value of a finitely supported symmetric a random walk $W_n = (z_n, f_n)$ on $\mathbb{Z} \wr \mathbb{Z}$, obtained after τ_n steps, where τ_n is the number of visits of $(\underbrace{e, e, \dots, e}_{j-1})$. Arguing by induction on j , we see that the expectation of τ_n

is $Cn^{1/2^{j-1}}$ for some $C > 0$. Fix some positive a such that $a < n^{1/2^{j+1}}$. Analogously to the case $j = 1$ we conclude that with probability close to one, for any sufficiently large n , there exists some integer y such that $|y - x_n| < n^a$ and such that $\phi_{j,n}(y) = f_n(\underbrace{e, e, \dots, e}_{j-1}, y) \geq C'n^{1/2^{j+1}}\sqrt{\ln(n)}$.

Take $C(n)$ tending to infinity and some $c' > 0$. Consider $V_n \subset G$ such that $\mu^{*n}(V_n) > c'$. Observe that there exist $g_n = (x_n, f_n) \in V_n$ and y_n such that $|y - x_n| < n^a$ and such that $f_n(\underbrace{e, e, \dots, e}_{j-1}, y) \geq C'n^{1/2^{j+1}}\sqrt{\ln(n)}$.

Now consider a $h_n = h_{n,j} \in G_j$, defined recursively by $h_{n,1} = (e, f')$, where $f'_1(y_n) = f'(y_n) = [-C(n)n^{1/2^{j+1}}/\sqrt{\ln(n)}]$ and $f'(x) = 0$ for all $x \neq y_n$; and $h_{n,k+1} = (e, f'_{k+1})$, where $f'_{k+1}(0) = f'_k$ and $f'(x) = 0$ for all $x \neq 0$. It is clear that the length of $h_{n,j}$ is at most $2C(n)n^{1/2^{j+1}}/\sqrt{\ln(n)}$. Observe that for any $K > 0$ and any sufficiently large n it holds $\mu_j^{*n}(g_n h_n)/\mu_j^{*n}(g_n) \geq K$.

This shows that the random walk (G_j, μ_j) is not almost invariant on the scale $C(n)n^{1/2^{j+1}}/\sqrt{\ln(n)}$ (for none of the the function $C(n)$ tending to infinity). This implies that the almost invariance radius for the random walk (G_j, μ_j) is not greater than $n^{1/2^{j+1}}/\sqrt{\ln(n)}$, and completes the proof of the proposition.

4.1. Wreath product of \mathbb{Z}^2 with finite groups. Given a random walk on (\mathbb{Z}^2, ν) , denote by $r_{\text{cov}}(n) = r_{\text{cov}}(\nu, n)$ the function such that for any ϵ there exists c with the following property. For all sufficiently large n the set of points of \mathbb{Z}^2 visited at least once until the moment n contains the ball of radius $cr_{\text{cov}}(n)$ with probability at least $1 - \epsilon$.

propositionwrz2

Proposition 4.7. *Let F be a finite group containing at least two elements. Let μ be a measure on $G = \mathbb{Z}^2 \wr F$ which defines a standard aperiodic switch-walk-switch random walk. Denote by ν the projection of μ to \mathbb{Z}^2 . We assume that the support of ν consists of 5 elements: 2 standard generators of \mathbb{Z}^2 , their inverses and the neutral element. Then the radius of individual almost invariance of (G, μ) satisfies $\bar{r}(n) \geq r_{\text{cov}}(\nu, n)$.*

Proof. Let S denote some finite generating set of G .

We want to show that for any $r(n) : r(n)/r_{\text{cov}}(\nu, n) \rightarrow 0$ there exists $V_n \subset G$ such that $\mu^{*n}(V_n) \rightarrow 1$ and such that for any $g_n \in V_n$ and any $h_n \in G$ such that $l_S(h_n) \leq r(n)$ it holds $\mu^{*n}(g_n h_n)/\mu^{*n}(g_n) \rightarrow 1$ as $n \rightarrow \infty$.

Analogously to the proof of Theorem I.3, it is sufficient to prove this statement separately for h_n having trivial projection to \mathbb{Z}^2 (h_n belongs to $\sum_{\mathbb{Z}^2} F$) and $h_n \in \mathbb{Z}^2$.

First observe that $r_{\text{cov}}(\nu, n) \leq \sqrt{n}$ since the number of points, visited until the moment n is at most n .

Take any function $r'(n)$ such that $r(n)/r'(n)$ tends to 0. Put $r''(n) = \sqrt{r_{\text{cov}}(n)r'(n)}$. We have $r(n) \leq r'(n) \leq r''(n) \leq r_{\text{cov}}$ and $(r')^2/n, (r'')^2/n \rightarrow 0$ as $n \rightarrow \infty$. Observe that $r''(n)/r_{\text{cov}}(n) \rightarrow 0$ and $r'(n)/r''(n) \rightarrow 0$. Denote by π the quotient map from G to \mathbb{Z}^2 .

Consider the event α_n on the space of trajectories of length n for the random walk (G, μ) that consists of the trajectories $X_1, X_2, \dots, X_n, X_i = (x_i, f_i)$ such that

- (1) the projection of the trajectory to \mathbb{Z}^2 visits all points of the ball of radius $2r''(n)$, centered at the origin, until the moment n ;
- (2) the projection of the trajectory to \mathbb{Z}^2 visits all points of the ball of radius $2r''(n)$, centered at x_n , until the moment n ;
- (3) Moreover, put $T_n = (r'(n))^2$. The projection of the trajectory to \mathbb{Z}^2 visits all points of the ball of radius $2r''(n - T_n)$, centered at x_{n-T_n} , until the moment $n - T_n$ and $|x_{n-T_n}, x_n| \leq r''(n)$.

Observe that (3) implies in particular, that the trajectory of the random walk visits until the moment $n - T_n$ all the points in the ball of radius $r''(n)$, centered at X_n .

Denote by μ_n the conditional measure $\mu^{*n}|\alpha_n$.

First let us show probability of α_n tends to 1 as $n \rightarrow \infty$. Indeed, the property (1) holds with probability close to one, as it follows from the definition of r_{cov} and from the fact that $2r''(n)/r_{\text{cov}}(n) \rightarrow 0$ as $n \rightarrow \infty$. To prove that (2) holds with probability close to one, consider the "inverted trajectory" of length n : $e, (X_n X_{(n-1)}^{-1})^{-1}, (X_n X_{(n-1)}^{-1})^{-1} (X_{n-1} X_{(n-2)}^{-1})^{-1} \dots$ and observe that the trajectory $e, X_1, \dots, X(n)$ visits the ball of radius $f(n)$ centered at X_n if and only the inverted trajectory visits the ball of the same radius, centered at the origin. The first property of (3) holds with probability close to one, as follows from (2) applied to $n' = n - T_n$. The second property of (3) holds since $r''(n)/\sqrt{T_n} \rightarrow \infty$ as $n \rightarrow \infty$.

Now take h_n such that $l_S(h_n) \leq r(n)$. First suppose that $h_n \in \sum_{\mathbb{Z}^2} F$. Using $r'(n) \leq r''(n)$ and (1) in the definition of α_n we observe that for any $g \in G$ $\mu_n(h_n g|\alpha_n) = \mu_n(g|\alpha_n)$. Since $\mu_n(\alpha_n) \rightarrow 1$ as $n \rightarrow \infty$, this implies that for some sets V_n such that $\mu_n(V_n) \rightarrow 1$ it holds $\mu_n(h_n g)/\mu_n(g)$ is close to one for any $g \in V_n$ and any h_n such that $l_S(h_n) \leq r'(n)$. Since μ is symmetric, this also implies that for some V_n such that $\mu_n(V_n) \rightarrow 1$, any $g \in V_n$ and any h_n such that $l_S(h_n) \leq r'(n)$ it holds $\mu_n(g h_n)/\mu_n(g)$ is close to one.

Now suppose that $h_n \in \mathbb{Z}^2$, $l(h_n) \leq r(n)$. For the projection of the random walk to \mathbb{Z}^2 , consider the conditional event corresponding to the properties 1) -3). Let ν_n be the corresponding conditional measure. Observe that for n tending to infinity

$$P_{\nu_n}[x_{n-T_n} = x', x_n = x'']/P_{\nu_n}[x_{n-T_n} = x', x_n = x'' h_n] \rightarrow 1$$

Therefore, for some set V_n such that $\mu^{*n}(V_n) \rightarrow 1$, any sufficiently large n and any $g \in V_n$ the ratio $\mu_n(g h_n)/\mu_n(g)$ is close to one.

Now we apply 2) of Lemma 2.12. We observe that for any $h_n \in G$ such that $l(h_n) \leq r(n)$ there exist V_n such that $\mu^{*n}(V_n) \rightarrow 1$, such that for any $g_n \in V_n$ it holds $\mu^{*n}(g_n h_n) / \mu^{*n}(g) \rightarrow 1$ as n tends to ∞ .

This completes the proof of Proposition 4.7.

Corollary 4.8. *Under the assumption of Proposition 4.7, the radius of individual almost invariance of the simple random walk (G, μ) is asymptotically strictly larger than $n/L(n)$. Here $L(n)$ denotes as before the drift function of the random walk (G, μ) .*

Proof. It is sufficient to use the estimate of covered balls, centered at the origin in \mathbb{Z}^2 due to Révész (see e.g. Theorem 24.2 [21], Chapter 24) that states that for some $C > 0$

$$\liminf_{n \rightarrow \infty} P[(\ln(R_{\text{cov}}(n))^2 / \log(n) > z] \geq \exp(-Cz)$$

(A precise limit distribution (of the constant above) for size of the ball in \mathbb{Z}^2 , centered at the origin, and covered by the trajectory of length n is due to Dembo, Peres, Rosen and Zeitouni ([3]).

Proposition 4.7 implies therefore that a random walk on $\mathbb{Z}^2 \wr F$ is individually almost invariant on the scale $f(n)$, for any $f(n)$ such that $(\ln f(n))^2 / \ln(n) \rightarrow 0$ as $n \rightarrow \infty$. In other words, for any ϵ_n , tending to 0 as $n \rightarrow \infty$, the simple random walk on $\mathbb{Z}^2 \wr F$ is individually almost invariant on the scale $\exp(\epsilon_n \sqrt{\ln(n)})$. Therefore, the radius of individual almost invariance of this random walk is greater than $\exp(\sqrt{\ln(n)})$. In particular, this radius is strictly larger asymptotically than $\ln(n)$.

The Corollary follows therefore from the fact that $L(n) \sim n/\ln(n)$ for any simple random walk on $\mathbb{Z}^2 \wr F$ (see [5] for a switch walk switch random walk, and a similar argument works for any simple random walk on this group).

piecewise

5. TRANSITION PROBABILITIES OF PIECEWISE AUTOMATIC GROUPS

The goal of this subsection is to prove Proposition I.8

Given a function $F(n)$, we need to construct a group G such that for all ϵ, K the distributions μ^{n_i} of simple random walks on G are not (ϵ, K) almost invariant with respect to balls of radius $F(n)$, for none of the choice of the subsequence n_i .

We will use the construction from [8].

In this paper, we define and study "piecewise automatic groups", and we show (see Proposition 1 in [8] and the proof of this proposition) that for τ_1 being the standard finite state automaton for the first Grigorchuk group (possibly extended to some larger alphabet than 0 and 1) and τ_2 being a finite state automata, containing e, a, b, c, d as its states, such that a, b, c, d generate the free product $\mathbb{Z}/2\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z})$, then the corresponding "piecewise automatic groups with returns" has the following properties.

For any $t_1 < T_1 < t_2 < T_2 < t_3 \dots$ one constructs a group $G(t_1, t_2, \dots, T_1, T_2, \dots)$ with the following properties. There exist "comparison groups" $A(t_1, T_1, t_2, T_2, \dots, t_i)$ and $B(t_1, T_1, t_2, T_2, \dots, t_i, T_i)$, such that

- (1) all groups $A(t_1, T_1, t_2, T_2, \dots, t_i)$ are commensurable with a group which imbeds as a subgroup in a finite direct power of the the first Grigorchuk group G_1 ,
- (2) all groups $B(t_1, T_1, t_2, T_2, \dots, t_i, T_i)$ are commensurable with a group which surjects to the group G_2 , generated by the automaton τ_2 ,
- (3) furthermore, for some $\Psi(x)$ tending to ∞ as x tends to ∞ the balls of radius $\Psi(T_i)$ in $G(t_1, t_2, \dots, T_1, T_2, \dots)$ and $A(t_1, T_1, t_2, T_2, \dots, t_i)$ coincide,
- (4) the balls of radius $\Psi(r_{i+1})$ in $G(t_1, t_2, \dots, T_1, T_2, \dots)$ and $B(t_1, T_1, t_2, T_2, \dots, t_i, T_i)$ coincide.

In particular, these properties imply that the constructed groups are of sub-exponential growth, so far as the sequence $t_1 < T_1 < t_2 < T_2 < t_3 \dots$ grows quickly enough. In particular, symmetric finitely supported random walks on these group have trivial Poisson boundary. By Proposition 2.6 we know that in this case the group admits a non-trivial radius of almost invariance.

Observe that the groups $B(t_1, T_1, t_2, T_2, \dots, t_i, T_i)$ are non-amenable, for any quickly enough growing sequence $t_1 < T_1 < t_2 < T_2 < t_3 < \dots$. This implies that any non-degenerate random walk on these groups has non-trivial Poisson boundary. By Proposition 2.6 we know that for any K, ϵ there exists C (depending on $B(t_1, T_1, t_2, T_2, \dots, t_i, T_i)$) such that the random walk on $B(t_1, T_1, t_2, T_2, \dots, t_i, T_i)$ is not (K, ϵ) on the balls of radius C .

The remark 5.1 below implies therefore that for any $F(n)$, tending to ∞ there exists a sequence $t_1 < T_1 < t_2 < T_2 < t_3 < \dots$ such that the corresponding piecewise automatic group satisfies the following property. For any K, ϵ and any sufficient large n any almost invariance radius $r_{\epsilon, K}(n) \leq F(n)$, and this completes the proof of the Proposition 1.8.

remarkCayley

Remark 5.1. Let A and B the groups generated by S_A and S_B such that their marked Cayley graphs are isometric in the balls of radius $R \geq 1$. Let μ_A and μ_B be probability measures supported on S_A and S_B such that $\pi(\mu_A) = \mu_B$, for isomorphism π between the balls of Cayley graphs (e.g. μ_A is an equidistribution on S_A and μ_B is an equidistribution on S_B). If the distributions of (A, μ_A) are (K, ϵ) almost invariant for $g \in A$ such that $l_{S_A} \leq r(n)$ and all n , then for any n such that $n + r(n) \leq R$ the distributions of (B, μ_B) are (K, ϵ) almost invariant for $g \in B$ such that $l_{S_B} \leq r(n)$.

Proof. Observe that for any n such that $n + r(n) \leq R$, any $h \in A$ in the support of μ_A^{*n} and any $g \in A$ such that $l_A(g) \leq r(n)$, it holds $l_A(gh) \leq R$.

6. NON CONSTANT LIMIT DISTRIBUTIONS FOR RATIOS OF TRANSITION PROBABILITIES

6.1. Limit distributions for word lengths. Given a finitely generated group G and a finite set of generators S of G , we consider the word length l_S . Given a sequence $g_n \in G$ (or a sequence of subsets $V_n \subset G$ and a sequence μ_n of probability measures on G , we can ask what is the limit behavior of $l_S(xg_n) - l_S(x)$ (where g_n is a fixed sequence or $g_n \in V_n$), considered as a function on the probability space (G, μ_n) . As before, we are interested in this paper in the case where $\mu_n = \mu^{*n}$ for some probability measure μ on G .

Lemma 6.1. [Free groups, distributions of $l_S(xh) - l_S(x)$] Let S be the free generating set of $G = F_m$, and μ be a measure which is equidistributed on $S \cup S^{-1}$. Fix an element $h \in F_m$.

There exists ϵ_n , tending to 0 as $n \rightarrow \infty$ such that for any $h \in G$ and any $k \leq l_S(h)$ it holds $|\mu^{*n}(g : l_S(gh) < l_S(g) + l(h) - 2k) - 1/v_m(k)| \leq \epsilon_n$.

Proof. Let $l_S(g) = L$ and $l_S(h) = K$. Write $g = s_{i_1} \dots s_{i_L}$, $s_j \in S$ and $h = t_{j_1} \dots t_{j_K}$, $t_j \in S$. If $t_{j_1} \neq s_{i_L}^{-1}$, then $l_S(gh) = K + L$. If $t_{j_1} = s_{i_L}^{-1}$ but $t_{j_2} \neq s_{i_{L-1}}^{-1}$, then $l_S(gh) = K + L - 2$. If $t_{j_1} = s_{i_L}^{-1}$, $t_{j_2} = s_{i_{L-1}}^{-1}$, but $t_{j_3} \neq s_{i_{L-2}}^{-1}$ then $l_S(gh) = K + L - 3$ etc

Observe that $\mu^{*n}(x) = \mu^{*n}(y)$ for any $x, y : l_S(x) = l_S(y)$.

Since the random walk (G, μ) is transient, we know that for any $C > 0$ it holds $\mu^{*n}(B(e, C)) \rightarrow 0$, where $B(e, C)$ is the ball of radius C in the word length l_S .

Fix an element h of length L . Observe that if we chose an element g at random from the sphere of radius $K > L$ in (G, l_S) , then with probability $1 - 1/v_m(1)$ the last letter $s_{i_L}^{-1}$ of

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g is not equal to the inverse of the first letter $t_{j_1}^{-1}$ of h . In this case $l_S(gh) = l_S(g) + l(h)$. With probability $1 - 1/v_m(2)$ at least one two among the two pairs of letters are not equal: either $t_{j_1} \neq s_{i_L}^{-1}$ or $t_{j_2} \neq s_{i_{L-1}}^{-1}$. In this case $l_S(gh) \geq l_S(g) + l(h) - 2$. For any $r \geq 1$ the following holds: at least m among the letters of g and the corresponding inversed letters of h are not equal with probability $1 - 1/v_m(r)$, and in this case $l_S(gh) \geq l_S(g) + l(h) - 2r$.

To prove the statement of the Lemma it suffices therefore to consider $\epsilon_n = \mu^{*n}(B(e, K))$.

6.2. Non-constant limit distributions. Though no group with non-trivial boundary can admit a non-trivial scale for almost invariance, it may happen that such groups admit a scale of weak almost constant contraction. That is, a scale on which the multiplication by an element g multiplies the transition probability by some constant C_g , up to a multiplicative constant and with probability close to one. This is the case for simple random walk on a free group F_m .

Let $v_m(n) = (2m - 1)(2m)^{n-1}$ be the spherical growth function of the free group. with respect to the free generating set.

examplefree

Example 6.2. Let S be the free generating set of F_m , $m \geq 2$, and μ be a measure which is equidistributed on $S \cup S^{-1}$.

Let ν_m be the probability measure on the set $0, 1, 2, \dots$ such that $\nu_m(-x) = 1/v_m(x) - 1/v_m(x+1) = 1/2m(1/(2m-1)^x - 1/(2m-1)^{x+1})$ for any integer $x \geq 0$. Consider the function $f(x) = -2x$ on this probability space. Take any sequence $g_n \in F_m$ such that $l(g_n) \rightarrow \infty$ and $l(g_n)/\sqrt{n} \rightarrow 0$, where l is some word metric on F_m . Then the distributions of the function $\mu^{*n}(g)/\mu^{*n}(gg_n)$ on the probability space μ^{*n} tend to that of f on ν_m .

Proof. Consider the random walk X_n on \mathbb{Z}^+ , reflected at zero, which goes with probability $1/(2m)$ to the left and $(2m-1)/2m$ to the right for any $x \in \mathbb{Z}^+$, $x \neq 0$. Observe that the simple random walk on F_m visits an element of length l with probability of $X_n = l$, and that for two elements g and h in the free group of the same word length with respect to S it holds $\mu^{*n}(g) = \mu^{*n}(h)$ for all n .

In particular,

$$\mu^{*n}(g) = 1/v(n)P[X_n = l_S(g)].$$

Remark 6.3. Let $X_n^o \in \mathbb{Z}^+$ be such that $(X_n^o - (m-1)/mn)/\sqrt{n} \rightarrow 0$ and such that $(X_n^o - [(m-1)/mn])$ is divided by two. Then $P[X_n = X_n^o]/P[X_n = [(m-1)/mn]] \rightarrow 1$. Otherwise, $(X_n^o - [(m-1)/mn])$ is not divided by two for each n , then $P[X_n = X_n^o]/P[X_n = [(m-1)/mn] + 1] \rightarrow 1$.

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The statement of Example 6.2 follows therefore from the Lemma 6.1.

REFERENCES

- [1] André Avez, *Limite de quotients pour des marches aléatoires sur des groupes*, C. R. Acad. Sci. Paris Sér. A-B **276** (1973), A317–A320 (French). MR0315750 (47 #4299)
- [2] Yves de Cornulier, Romain Tessera, and Alain Valette, *Isometric group actions on Hilbert spaces: growth of cocycles*, Geom. Funct. Anal. **17** (2007), no. 3, 770–792, DOI 10.1007/s00039-007-0604-0. MR2346274 (2009g:22007)
- [3] Amir Dembo, Yuval Peres, Jay Rosen, and Ofer Zeitouni, *Cover times for Brownian motion and random walks in two dimensions*, Ann. of Math. (2) **160** (2004), no. 2, 433–464, DOI 10.4007/annals.2004.160.433. MR2123929
- [4] P. Erdős and M. Kac, *On certain limit theorems of the theory of probability*, Bull. Amer. Math. Soc. **52** (1946), 292–302. MR0015705 (7,459b)
- [5] A. G. Dyubina, *An example of the rate of departure to infinity for a random walk on a group*, Uspekhi Mat. Nauk **54** (1999), no. 5(329), 159–160, DOI 10.1070/rm1999v054n05ABEH000208 (Russian); English transl., Russian Math. Surveys **54** (1999), no. 5, 1023–1024. MR1741670

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- [6] Anna Erschler, *On drift and entropy growth for random walks on groups*, Ann. Probab. **31** (2003), no. 3, 1193–1204, DOI 10.1214/aop/1055425775. MR1988468 (2004c:60018)
- [7] ———, *Critical constants for recurrence of random walks on G -spaces*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 2, 493–509. MR2147898 (2006c:20085)
- [8] ———, *Piecewise automatic groups*, Duke Math. J. **134** (2006), no. 3, 591–613, DOI 10.1215/S0012-7094-06-13435-X. MR2254627 (2007k:20086)
- [9] Anna Erschler and Anders Karlsson, *Homomorphisms to \mathbb{R} constructed from random walks*, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 6, 2095–2113 (English, with English and French summaries). MR2791651 (2012c:60018)
- [10] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985 (Russian). MR764305 (86h:20041)
- [11] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. **53** (1981), 53–73. MR623534 (83b:53041)
- [12] W. Hebisch and L. Saloff-Coste, *Gaussian estimates for Markov chains and random walks on groups*, Ann. Probab. **21** (1993), no. 2, 673–709. MR1217561 (94m:60144)
- [13] V. A. Kaimanovich and A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983), no. 3, 457–490. MR704539 (85d:60024)
- [14] Vadim A. Kaimanovich, *Measure-theoretic boundaries of Markov chains, 0-2 laws and entropy*, Harmonic analysis and discrete potential theory (Frascati, 1991), Plenum, New York, 1992, pp. 145–180. MR1222456 (94h:60099)
- [15] Bruce Kleiner, *A new proof of Gromov’s theorem on groups of polynomial growth*, J. Amer. Math. Soc. **23** (2010), no. 3, 815–829, DOI 10.1090/S0894-0347-09-00658-4. MR2629989
- [16] Nicholas J. Korevaar and Richard M. Schoen, *Global existence theorems for harmonic maps to non-locally compact spaces*, Comm. Anal. Geom. **5** (1997), no. 2, 333–387. MR1483983 (99b:58061)
- [17] Gregory F. Lawler and Vlada Limic, *Random walk: a modern introduction*, Cambridge Studies in Advanced Mathematics, vol. 123, Cambridge University Press, Cambridge, 2010. MR2677157
- [18] Ngaiming Mok, *Harmonic forms with values in locally constant Hilbert bundles*, Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993), 1995, pp. 433–453. MR1364901 (97c:58008)
- [19] Narutaka Ozawa, *A functional analysis proof of Gromov’s polynomial growth theorem*, <http://arxiv.org/abs/1510.04223>.
- [20] David Revelle, *Heat kernel asymptotics on the lamplighter group*, Electron. Comm. Probab. **8** (2003), 142–154 (electronic), DOI 10.1214/ECP.v8-1092. MR2042753 (2005e:60100)
- [21] Pál Révész, *Random walk in random and non-random environments*, 2nd ed., World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. MR2168855 (2006e:60003)
- [22] Yehuda Shalom, *Harmonic analysis, cohomology, and the large-scale geometry of amenable groups*, Acta Math. **192** (2004), no. 2, 119–185, DOI 10.1007/BF02392739. MR2096453 (2005m:20095)
- [23] Frank Spitzer, *Principles of random walk*, 2nd ed., Springer-Verlag, New York-Heidelberg, 1976. Graduate Texts in Mathematics, Vol. 34. MR0388547 (52 #9383)